$$\mathbf{I} = \int \left(x^8+1\right)^{-1} dx$$

In the moment when I truly understand my enemy, understand him well enough to defeat him, then in that very moment I also love him. I think it's impossible to really understand somebody, what they want, what they believe, and not love them the way they love themselves.

And then, in that very moment when I love them... I destroy them.

– *Ender's Game*

---

The first thing to note is that the denominator indeed factors:

$$x^8+1 = \left(x^2\right)^4+1 = \left(x^4-\sqrt{2}x^2+1\right)\left(x^4+\sqrt{2}x^2+1\right).$$

This allows us to do partial fractions.

$$\frac{Ax^2+B}{x^4-\sqrt{2}x^2+1}+\frac{Cx^2+D}{x^4+\sqrt{2}x^2+1} = \frac{\left(x^4+\sqrt{2}x^2+1\right)\left(Ax^2+B\right)+\left(x^4-\sqrt{2}x^2+1\right)\left(Cx^2+D\right)}{x^8+1}$$

The numerator expands as:

$$(A+C)x^6+\left(\sqrt{2}(A-C)+(B+D)\right)x^4+\left((A+C)+\sqrt{2}(B-D)\right)x^2+(B+D)$$

Setting $C=-A$ and $B+D=1$, this simplifies to:

$$\left(2\sqrt{2}A+1\right)x^4+\sqrt{2}(B-D)x^2+1.$$

Therefore,

$$A=-\frac{1}{2\sqrt{2}}, \qquad B=\frac{1}{2}, \qquad C=\frac{1}{2\sqrt{2}}, \qquad D=\frac{1}{2}.$$

Substituting and simplifying, we obtain:

$$\mathbf{I} = \frac{1}{2\sqrt{2}}\int\left(\frac{x^2+\sqrt{2}}{x^4+\sqrt{2}x^2+1}-\frac{x^2-\sqrt{2}}{x^4-\sqrt{2}x^2+1}\right)dx.$$

---

Now let

$$J = \int \frac{x^2+\alpha}{x^4+\alpha x^2+1}dx.$$

Again we may factor the denominator:

$$x^4+\alpha x^2+1 = \left(x^2-\sqrt{2-\alpha}\,x+1\right)\left(x^2+\sqrt{2-\alpha}\,x+1\right).$$

Now back to partial fractions.

$$\frac{Ax+B}{x^2-\sqrt{2-\alpha}\,x+1}+\frac{Cx+D}{x^2+\sqrt{2-\alpha}\,x+1} = \frac{\left(x^2+\sqrt{2-\alpha}\,x+1\right)(Ax+B)+\left(x^2-\sqrt{2-\alpha}\,x+1\right)(Cx+D)}{x^4+\alpha x^2+1}$$

The numerator expands as:

$$(A+C)x^3+\left(\sqrt{2-\alpha}\,(A-C)+(B+D)\right)x^2+\left(\sqrt{2-\alpha}\,(B-D)+(A+C)\right)x+(B+D)$$

Setting $C=-A$ and $B+D=\alpha$ yields

$$\left(2A\sqrt{2-\alpha}+\alpha\right)x^2+\left(\sqrt{2-\alpha}(B-D)\right)x+\alpha.$$

Therefore,

$$A = \frac{1-\alpha}{2\sqrt{2-\alpha}}, \qquad B = \frac{\alpha}{2}, \qquad C = \frac{\alpha-1}{2\sqrt{2-\alpha}}, \qquad D = \frac{\alpha}{2}.$$

Substituting and simplifying, we obtain:

$$J = \left(\frac{1-\alpha}{2\sqrt{2-\alpha}}\right)\int\left(\frac{x+\frac{\alpha\sqrt{2-\alpha}}{1-\alpha}}{x^2-\sqrt{2-\alpha}\,x+1} - \frac{x-\frac{\alpha\sqrt{2-\alpha}}{1-\alpha}}{x^2+\sqrt{2-\alpha}\,x+1}\right)dx.$$

Set $\beta = \sqrt{2-\alpha}$ so that $\alpha = 2-\beta^2$. Then

$$\frac{\alpha\sqrt{2-\alpha}}{1-\alpha} = \frac{(2-\beta^2)\beta}{1-(2-\beta^2)} = \frac{2\beta-\beta^3}{\beta^2-1}, \qquad \frac{1-\alpha}{2\sqrt{2-\alpha}} = \frac{\beta^2-1}{2\beta}$$

Our new expression for $J$ is thus:

$$J = \frac{\beta^2-1}{2\beta}\int\left(\frac{x+\frac{2\beta-\beta^3}{\beta^2-1}}{x^2-\beta x+1} - \frac{x-\frac{2\beta-\beta^3}{\beta^2-1}}{x^2+\beta x+1}\right)dx.$$

Now note that

$$x+\frac{2\beta-\beta^3}{\beta^2-1} = \frac{1}{2}(2x-\beta)+\left(\frac{\beta}{2}+\frac{2\beta-\beta^3}{\beta^2-1}\right), \qquad x-\frac{2\beta-\beta^3}{\beta^2-1} = \frac{1}{2}(2x+\beta)-\left(\frac{\beta}{2}+\frac{2\beta-\beta^3}{\beta^2-1}\right),$$

which implies

$$J = \frac{\beta^2-1}{4\beta}\int\left(\frac{2x-\beta}{x^2-\beta x+1}+\frac{\frac{\beta}{2}+\frac{2\beta-\beta^3}{\beta^2-1}}{\left(x-\frac{\beta}{2}\right)^2+1-\frac{\beta^2}{4}} - \frac{2x+\beta}{x^2+\beta x+1}+\frac{\frac{\beta}{2}+\frac{2\beta-\beta^3}{\beta^2-1}}{\left(x+\frac{\beta}{2}\right)^2+1-\frac{\beta^2}{4}}\right)dx$$

$$= \frac{\beta^2-1}{4\beta}\left(\log\left(\frac{x^2-\beta x+1}{x^2+\beta x+1}\right)+\left(\frac{\beta}{2}+\frac{2\beta-\beta^3}{\beta^2-1}\right)\left(1-\frac{\beta^2}{4}\right)\left(\arctan\left(\frac{x-\frac{\beta}{2}}{\sqrt{1-\frac{\beta^2}{4}}}\right)+\arctan\left(\frac{x+\frac{\beta}{2}}{\sqrt{1-\frac{\beta^2}{4}}}\right)\right)\right)+C$$

$$= \frac{1-\alpha}{4\sqrt{2-\alpha}}\log\left(\frac{x^2-\sqrt{2-\alpha}\,x+1}{x^2+\sqrt{2-\alpha}\,x+1}\right)+\frac{(1+\alpha)(2+\alpha)}{32}\left(\arctan\left(\frac{2x-\sqrt{2-\alpha}}{\sqrt{\alpha+2}}\right)+\arctan\left(\frac{2x+\sqrt{2-\alpha}}{\sqrt{\alpha+2}}\right)\right)+C$$

In particular, for $\alpha = \sqrt{2}$,

$$J = \frac{1-\sqrt{2}}{4\sqrt{2-\sqrt{2}}}\log\left(\frac{x^2-\sqrt{2-\sqrt{2}}\,x+1}{x^2+\sqrt{2-\sqrt{2}}\,x+1}\right)+\frac{(1+\sqrt{2})(2+\sqrt{2})}{32}\left(\arctan\left(\frac{2x-\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)+\arctan\left(\frac{2x+\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)\right)+C$$

and for $\alpha = -\sqrt{2}$,

$$J = \frac{1+\sqrt{2}}{4\sqrt{2+\sqrt{2}}}\log\left(\frac{x^2-\sqrt{2+\sqrt{2}}\,x+1}{x^2+\sqrt{2+\sqrt{2}}\,x+1}\right)+\frac{(1-\sqrt{2})(2-\sqrt{2})}{32}\left(\arctan\left(\frac{2x-\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)+\arctan\left(\frac{2x+\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)\right)+C$$

---

Concluding,

$$\boxed{\begin{aligned} I = {}& \frac{1}{2\sqrt{2}}\left(\frac{1-\sqrt{2}}{4\sqrt{2-\sqrt{2}}}\log\left(\frac{x^2-\sqrt{2-\sqrt{2}}\,x+1}{x^2+\sqrt{2-\sqrt{2}}\,x+1}\right)+\frac{(1+\sqrt{2})(2+\sqrt{2})}{32}\left(\arctan\left(\frac{2x-\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)+\arctan\left(\frac{2x+\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)\right)\right) \\ & -\frac{1}{2\sqrt{2}}\left(\frac{1+\sqrt{2}}{4\sqrt{2+\sqrt{2}}}\log\left(\frac{x^2-\sqrt{2+\sqrt{2}}\,x+1}{x^2+\sqrt{2+\sqrt{2}}\,x+1}\right)+\frac{(1-\sqrt{2})(2-\sqrt{2})}{32}\left(\arctan\left(\frac{2x-\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)+\arctan\left(\frac{2x+\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)\right)\right)+C. \end{aligned}}$$