Finite $\implies$ (Injective $\iff$ Surjective)

Abstract

This writeup establishes two facts:

1. If $f : S \to S$ where $|S| < \infty$, then f injects iff f surjects.
2. If $T : V \to V$ where $\dim V < \infty$, then T injects iff T surjects.

These are basically the same thing; we hope to make this clear by juxtaposing the arguments.

Finite Sets: Injective $\implies$ Surjective

Let S be a nonempty finite set with N elements and let $f : S \to S$.

Claim: *If f is injective, then f is surjective.*

Proof. Suppose f is injective. Let $s \in S$, and consider the elements

$$s, \quad f(s), \quad f^2(s), \quad \ldots, \quad f^N(s)$$

in S. Since S only has N elements, we have

$$f^m(s) = f^M(s)$$

for some $0 \leq m \leq M$ (without loss of generality). But then by injectivity of f, we have

$$s = f(f^{M-m-1}(s)),$$

which shows that f is surjective. □

Finite Sets: Surjective $\implies$ Injective

Let S be a nonempty finite set with N elements and let $f : S \to S$.

Claim: *If f is surjective, then f is injective.*

Proof. Suppose f is surjective. Pick $a, b \in S$ and suppose

$$f(a) = f(b) =: c.$$

Then $S \setminus \{c\}$ has $N - 1$ elements, hence $f^{-1}(S \setminus \{c\}) = S \setminus f^{-1}(\{c\})$ has at least $N - 1$ elements, hence $f^{-1}(\{c\})$ has at most one element, hence $a = b$, which shows that f is injective. □

Finite Sets with Zero Elements

If $S = \varnothing$, then the only map from S to S is the null map, which consists of zero ordered pairs and is vacuously bijective.

Finite Dimensional Vector Spaces: Injective $\implies$ Surjective

Let V be a nontrivial vector space of dimension N, and let $T : V \to V$.

Claim: *If f is injective, then f is surjective.*

Proof. Suppose T is injective, i.e. suppose $\ker T = \{0\}$. Let $v \in V$.

Consider the vectors:

$$v, \quad Tv, \quad T^2v, \quad \ldots, \quad T^Nv.$$

Since V has dimension N, this is a linearly dependent set, i.e. there exists $\lambda = (\lambda_0, \ldots, \lambda_N) \neq 0$ such that

$$\lambda_0 v + \lambda_1 Tv + \cdots + \lambda_N T^N v = 0.$$

Suppose $\lambda_0 \neq 0$. Then we may divide by λ_0 and use linearity of T to obtain:

$$v = T\left(-\frac{1}{\lambda_0}(\lambda_1 v + \cdots + \lambda^N T^{N-1} v)\right).$$

If $\lambda_0 = 0$, then $\lambda_1 Tv + \cdots + \lambda_N T^N v = 0$, and by linearity of T, we get $T(\lambda_1 v + \cdots + \lambda_N T^{N-1} v) = 0$, i.e.

$$\lambda_1 v + \cdots + \lambda_N T^{N-1} v \in \ker T.$$

But since T is injective, this implies

$$\lambda_1 v + \cdots + \lambda_N T^{N-1} v = 0,$$

and so we're back where we started. Since $\lambda \neq 0$, we eventually encounter a nonzero λ_i, which we may divide by, and then use linearity of T as before. This proves T is surjective. □

Finite Dimensional Vector Spaces: Surjective $\implies$ Injective

Let V be a nontrivial vector space of dimension N, and let $T : V \to V$.

Claim: *If f is surjective, then f is injective.*

Proof. Suppose T is surjective, i.e. suppose $T(V) = V$. Let $v \in \ker T$. Then $\operatorname{Span}(v) \subset \ker T$, so there exists a unique surjective linear map

$$\widetilde{T} : V/\operatorname{Span}(v) \to V,$$

which implies $\dim(V/\operatorname{Span}(v)) \geq N$. But we also have

$$\dim(V/\operatorname{Span}(v)) = N - \dim(\operatorname{Span}(v)) \leq N,$$

so $V/\operatorname{Span}(v)$ has dimension N. This implies $\operatorname{Span}(v) = \{0\}$, hence $v = 0$. This proves T is injective. □

Vector Spaces of Dimension Zero

If $V = \{0\}$, the only map from V to V is the zero map, which sends 0 to 0 and is trivially bijective.