Single Variable Calculus: Integration Basics

Abstract

This is a simple tutorial on how to generate antiderivatives. We emphasize using the differential dx as a function, in order to make the classic techniques (substitution, integration by parts) more intuitive. Instead of drilling arduous trigonometric substitutions, we work with Euler's formula. The general theme is to emphasize **pattern recognition** over rote memorization.

Though not absolutely required, some familiarity with implicit differentiation may be helpful.

In this writeup, $\log x$ *denotes the natural logarithm.*

The key to understanding the basic integration techniques is to have a clear idea of where they come from. Recall the chain rule and the product rule:

$$\frac{d(g \circ f)}{dx}(x) = \frac{dg}{dx}(f(x)) \cdot \frac{df}{dx}(x)$$

$$\frac{d(fg)}{dx}(x) = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)$$

Integration via substitution is a matter of recognizing the chain rule in reverse. For example,

$$d(x^2) = 2x\,dx \implies \int 2x\,dx = \int d(x^2) = x^2 + C.$$

Here are a few more:

$$\int \frac{\sin x\,dx}{1+\cos^2 x} = -\int \frac{d(\cos x)}{1+\cos^2 x} = -\int \frac{du}{1+u^2} = -\arctan u + C_0 = \boxed{-\arctan(\cos x) + C}$$

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^{x^3}\,d(x^3) = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C_0 = \boxed{\frac{1}{3}e^{x^3} + C}$$

$$\int \frac{dx}{x\log x} = \int \frac{1}{\log x}d(\log x) = \int \frac{du}{u} = \log u + C_0 = \boxed{\log\log x + C}$$

$$\int \frac{x^3\,dx}{\sqrt{1+x^2}} = \frac{1}{2}\int \frac{x^2\,d(x^2)}{1+x^2} = \frac{1}{2}\int \frac{u\,du}{\sqrt{1+u}} = \frac{1}{2}\int \frac{y^2-1}{y}2y\,dy = \int y^2 - 1\,dy$$

$$= \frac{1}{3}y(y^2-3) + C_0 = \frac{1}{3}\sqrt{1+u}(u-2) + C_1 = \boxed{\frac{1}{3}\sqrt{1+x^2}(x^2-2) + C}$$

From this viewpoint, the game is to find instances of $d(f(x)) = f'(x)\,dx$.

Similarly, **integration by parts is a matter of recognizing the product rule in reverse.** For example,

$$\begin{aligned}\int x^2 \sin x \, dx &= -\int x^2 \, d(\cos x) = -\int \left(d\left(x^2 \cos x\right) - \cos x \, d(x^2)\right) \\ &= C_0 - x^2 \cos x + 2\int x \cos x \, dx = C_0 - x^2 \cos x + 2\int x \, d(\sin x) \\ &= C_0 - x^2 \cos x + 2\int \left(d(x \sin x) - \sin x \, dx\right) \\ &= C_1 - x^2 \cos x + 2x \sin x - 2\int \sin x \, dx \\ &= \boxed{2x \sin x - (x^2 - 2)\cos x + C}\end{aligned}$$

From this viewpoint, the game is to find instances of $f(dg) = d(fg) - g(df)$.

Finally, **instead of trigonometric substutions, recall Euler's formula:**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Since cosine is even and sine is odd, we also get:

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

After a bit of rearranging we get the following helpful identities:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Slightly more generally,

$$\cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2}, \qquad \sin(n\theta) = \frac{e^{in\theta} - e^{-in\theta}}{2i}.$$

Here is the technique in action (some algebra steps skipped):

$$\begin{aligned}\int \sin^3\theta \cos^2\theta \, d\theta &= -\frac{1}{32i}\int \left(e^{3i\theta} - e^{-3i\theta} - 3\left(e^{i\theta} - e^{-i\theta}\right)\right)\left(e^{2i\theta} + 2 + e^{-2i\theta}\right) \, d\theta \\ &= -\frac{1}{32i}\int \left(e^{5i\theta} - e^{-5i\theta} - \left(e^{3i\theta} - e^{-3i\theta}\right) - 2\left(e^{i\theta} - e^{-i\theta}\right)\right) \, d\theta \\ &= \frac{1}{16}\int \left(-\sin(5\theta) + \sin(3\theta) + 2\sin\theta\right) d\theta \\ &= \boxed{\frac{1}{80}\cos(5\theta) - \frac{1}{48}\cos(3\theta) - \frac{1}{8}\cos\theta + C}\end{aligned}$$

If you're skilled at algebra but not so much at trigonometry, Euler's formula is your friend.

It is sometimes remarked that differentiation is a skill, whereas integration is an art. Hopefully this writeup communicates that, in certain contexts, **art reduces to applying skill in a thoughtful way**.