Electrodynamics: Connection and Curvature

Abstract

We derive the differential form of Maxwell's equations carefully, as well as the matrix form of F.

This writeup uses the $(-,+,+,+)$ signature convention. Let $c = \mu_0 = \varepsilon_0 = 1$.

Maxwell's equations in vector form are then:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho & \qquad \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \\ \nabla \cdot \mathbf{B} &= 0 & \qquad \nabla \times \mathbf{B} &= 4\pi \mathbf{J} + \partial_t \mathbf{E} \end{aligned}$$

Where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and $\mathbf{J}$ the electric current.

In tensor notation, we have:

$$\begin{aligned} \partial_i E^i &= 4\pi J^0 & \qquad \varepsilon^{ijk}\partial_j E_k &= -\partial_0 B^i \\ \partial_i B^i &= 0 & \qquad \varepsilon^{ijk}\partial_j B_k &= 4\pi J^i + \partial_0 E^i \end{aligned}$$

The **electromagnetic 4–potential** A can be expressed as a vector:

$$A^\sharp = \phi e_0 + A^1 e_1 + A^2 e_2 + A^3 e_3 \quad \longleftrightarrow \quad A^\alpha = \phi e_0 + A^i e_i$$

We may flatten this to obtain a 1-form:

$$A^\flat = \phi\, dx^0 + A_1\, dx^1 + A_2\, dx^2 + A_3\, dx^3 \quad \longleftrightarrow \quad A_\alpha = \phi\, dx^0 + A_i dx^i$$

Taking the exterior derivative of this expression,

$$\begin{aligned} dA^\flat &= d\big(\phi\, dx^0 + A_1\, dx^1 + A_2\, dx^2 + A_3\, dx^3\big) \\ &= d\big(\phi\, dx^0\big) + d\big(A_1\, dx^1\big) + d\big(A_2\, dx^2\big) + d\big(A_3\, dx^3\big) \\ &= \partial_0\phi\, dx^0 \wedge dx^0 + \partial_1\phi\, dx^1 \wedge dx^0 + \partial_2\phi\, dx^2 \wedge dx^0 + \partial_3\phi\, dx^3 \wedge dx^0 \\ &+ \partial_0 A_1\, dx^0 \wedge dx^1 + \partial_1 A_1\, dx^1 \wedge dx^1 + \partial_2 A_1\, dx^2 \wedge dx^1 + \partial_3 A_1\, dx^3 \wedge dx^1 \\ &+ \partial_0 A_2\, dx^0 \wedge dx^2 + \partial_1 A_2\, dx^1 \wedge dx^2 + \partial_2 A_2\, dx^2 \wedge dx^2 + \partial_3 A_2\, dx^3 \wedge dx^2 \\ &+ \partial_0 A_3\, dx^0 \wedge dx^3 + \partial_1 A_3\, dx^1 \wedge dx^3 + \partial_2 A_3\, dx^2 \wedge dx^3 + \partial_3 A_3\, dx^3 \wedge dx^3 \\ &= (\partial_0 A_1 - \partial_1\phi)\ dx^0 \wedge dx^1 + (\partial_0 A_2 - \partial_2\phi)\ dx^0 \wedge dx^2 + (\partial_0 A_3 - \partial_3\phi)\ dx^0 \wedge dx^3 \\ &+ (\partial_2 A_3 - \partial_3 A_2)\ dx^2 \wedge dx^3 + (\partial_3 A_1 - \partial_1 A_3)\ dx^3 \wedge dx^1 + (\partial_1 A_2 - \partial_2 A_1)\ dx^1 \wedge dx^2 \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu)\ dx^\mu \wedge dx^\nu \end{aligned}$$

Expressing the (antisymmetric) **electromagnetic tensor / Faraday 2–form** F as

$$F^{\flat\flat} = (\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu \quad \longleftrightarrow \quad F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \otimes dx^\nu$$

we see $dA^\flat = F^{\flat\flat}$ and hence $dF^{\flat\flat} = d(dA^\flat) = 0$. This is commonly simplified to

$$\boxed{dF = 0.}$$

The fields $\mathbf{E}$ and $\mathbf{B}$ can be expressed in terms of the potentials ϕ (electric) and $\mathbf{A}$ (magnetic) as:

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

In tensor notation, this is:

$$E^i = -\partial^i A^0 - \partial_0 A^i, \qquad B^i = \varepsilon^{ijk}\partial_j A_k$$

Simplifying,

$$\begin{aligned} E^i &= -\partial^i A^0 - \partial_0 A^i = -\eta^{ii}\partial_i A^0 - \partial_0 A^i \\ &= -\partial_i A^0 - \partial_0 A^i = -\partial_i A_0 \eta^{00} - \partial_0 A_i \eta^{ii} \\ &= \partial_i A_0 - \partial_0 A_i = -F_{i0} \end{aligned}$$

$$B^i = \varepsilon^{ijk}\partial_j A_k = \partial_j A_k - \partial_k A_j = F_{jk}$$

so that in matrix form (using antisymmetry of F),

$$[F_{\mu\nu}] = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{bmatrix}.$$

Let's rewrite F as a 2–form with these new expressions:

$$\begin{aligned} F^{\flat\flat} = {} & E^1\, dx^1 \wedge dx^0 + E^2\, dx^2 \wedge dx^0 + E^3\, dx^3 \wedge dx^0 \\ & + B^1\, dx^2 \wedge dx^3 + B^2\, dx^3 \wedge dx^1 + B^3\, dx^1 \wedge dx^2 \end{aligned}$$

Taking the Hodge star, differentiating, and then taking the Hodge star once more,

$$\begin{aligned} \star F^{\flat\flat} &= \star(E^1\, dx^1 \wedge dx^0 + E^2\, dx^2 \wedge dx^0 + E^3\, dx^3 \wedge dx^0) \\ &\quad + \star(B^1\, dx^2 \wedge dx^3 + B^2\, dx^3 \wedge dx^1 + B^3\, dx^1 \wedge dx^2) \\ &= E^1\, dx^2 \wedge dx^3 + E^2\, dx^3 \wedge dx^1 + E^3\, dx^1 \wedge dx^2 \\ &\quad - B^1\, dx^1 \wedge dx^0 - B^2\, dx^2 \wedge dx^0 - B^3\, dx^3 \wedge dx^0 \end{aligned}$$

$$\begin{aligned} d(\star F^{\flat\flat}) &= \partial_0 E^1 dx^0 \wedge dx^2 \wedge dx^3 + \partial_1 E^1 dx^1 \wedge dx^2 \wedge dx^3 + \partial_0 E^2 dx^0 \wedge dx^3 \wedge dx^1 + \partial_2 E^2 dx^2 \wedge dx^3 \wedge dx^1 \\ &\quad + \partial_0 E^2 dx^0 \wedge dx^1 \wedge dx^2 + \partial_3 E^3 dx^3 \wedge dx^1 \wedge dx^2 - \partial_2 B^1 dx^2 \wedge dx^1 \wedge dx^0 - \partial_3 B^1 dx^3 \wedge dx^1 \wedge dx^0 \\ &\quad - \partial_3 B^2 dx^3 \wedge dx^2 \wedge dx^0 - \partial_1 B^2 dx^1 \wedge dx^2 \wedge dx^0 - \partial_1 B^3 dx^1 \wedge dx^3 \wedge dx^0 - \partial_2 B^3 dx^2 \wedge dx^3 \wedge dx^0 \\ &= (\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3)\, dx^1 \wedge dx^2 \wedge dx^3 + (\partial_0 E^1 + \partial_3 B^2 - \partial_2 B^3)\, dx^0 \wedge dx^2 \wedge dx^3 \\ &\quad + (\partial_0 E^2 + \partial_1 B^3 - \partial_3 B^1)\, dx^0 \wedge dx^3 \wedge dx^1 + (\partial_0 E^3 + \partial_2 B^1 - \partial_1 B^2)\, dx^0 \wedge dx^1 \wedge dx^2 \end{aligned}$$

$$\begin{aligned} \star d(\star F^{\flat\flat}) &= -(\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3)\, dx^0 + (\partial_0 E^1 + \partial_3 B^2 - \partial_2 B^3)\, dx^1 \\ &\quad + (\partial_0 E^2 + \partial_1 B^3 - \partial_3 B^1)\, dx^2 + (\partial_0 E^3 + \partial_2 B^1 - \partial_1 B^2)\, dx^3 \\ &= -\partial_i E^i\, dx^0 + (\partial_0 E^i - \varepsilon^{ijk}\partial_j B^k)\, dx^i = -\partial_i E^i\, dx^0 + (\partial_0 E^i - \varepsilon^{ijk}\partial_j B_k \eta^{kk})\, dx^i \\ &= -\partial_i E^i\, dx^0 + (\partial_0 E^i - \varepsilon^{ijk}\partial_j B_k)\, dx^i = -4\pi J^0\, dx^0 + 4\pi J^i\, dx^i = -4\pi J_0 \eta^{00} dx^0 + 4\pi J_i \eta^{ii}\, dx^i \\ &= 4\pi J_0\, dx^0 + 4\pi J_i\, dx^i = 4\pi J_\alpha dx^\alpha = 4\pi J^\flat. \end{aligned}$$

This is commonly simpified to $\star d(\star F) = 4\pi J$ or, using the codifferential $\delta(-) = \star d(\star(-))$,

$$\boxed{\delta F = 4\pi J.}$$