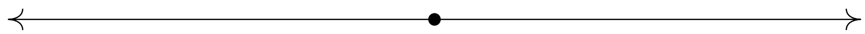
§ Introduction

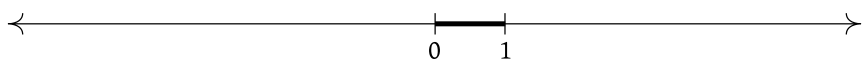
In mathematics, the simplest seeming ideas frequently draw forth entire worlds of complexity upon a closer look. For example, picture a line extending endlessly in both directions. Distinguish a point on the line to mark as center.



Claim: we may assign to each fraction a unique point on the line.

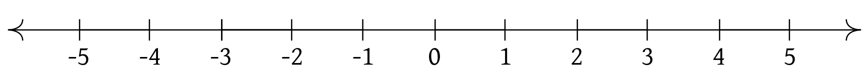
- Call the center point 0.
- The center point splits the line into two overlapping parts: one ray pointing left, one ray pointing right. The parts overlap at 0. Distinguish a point on the right ray excluding 0. Call it 1.

A segment now exists with endpoints 0 and 1. Call it the *unit interval.*



- Suppose the unit interval does not change length when we slide it. Slide the unit interval rightwards by one unit interval's length, so that 0 moves to 1 and 1 moves to a point yet unnamed. Call it 2. Continuing on in this way, we designate a unique point for each positive natural number.
- As the right end of the right ray has no endpoint, we never run out of space to place points, and so we may relate the endless list of positive natural numbers to a corresponding endless list of points on the right ray to the right of 0.
- Reflect the right ray about 0 so that the left ray now houses an endless list of points each at least unit distance apart, all to the left of 0. To each point n on the right ray excluding 0, assign to its reflection about 0 on the left the negative number $-n$.

The line now houses a mesh of points.



- We may shrink the mesh in a countless number of ways: each point n to the right of 1 represents a shrinking factor, wherein the point n maps to 1, the point 1 maps to the fraction $\frac{1}{n}$, and the rest of the mesh shrinks in proportion.
- Thus, to find the point associated with the fraction $\frac{p}{q}$, first start at 0, then move to the right q units, then shrink the mesh so that q ends up at 1, travel the length of the interval from 0 to $\frac{1}{q}$ a total of p times to the right, and then reflect about 0 if the fraction has a minus sign.

This shows the claim but raises a question: *may we assign to each point on the line a unique fraction?*

Before we begin, some conventions on notation:

- Read the symbol $\in$ as "in." Read the symbol $\notin$ as "not in."
- Denote by $\mathbf{Z}$ the set of integers (we shall say more about its structure later), formed by considering the natural numbers and their negatives as a whole. Denote by $\mathbf{N}$ the natural numbers.
- Denote by $\mathbf{Z}^+$ the positive integers (nonzero natural numbers), by $\mathbf{Z}^-$ the negative integers (negatives of positive integers).
- Denote by $2\mathbf{Z}$ the even integers (any $x \in \mathbf{Z}$ such that $x = 2 \cdot y$ for some $y \in \mathbf{Z}$).
- Denote by $\mathbf{Q}$ the set of fractions (we shall say more about its structure later). Denote by $\mathbf{Q}^+$ the positive fractions.

Existence of Irrational Numbers

THEOREM 0.1 If a number x has the property

$$x \cdot x = 2$$

then $x \notin \mathbf{Q}$.

PROOF. Suppose we have such an x, and suppose further that we may write it as a fraction $\frac{p}{q}$ (in lowest terms, so that p and q have no common factors). In other words, suppose $x \in \mathbf{Q}$. Then

$$2 = x \cdot x = \left(\frac{p}{q}\right) \cdot \left(\frac{p}{q}\right) = \frac{p \cdot p}{q \cdot q}.$$

Multiply both sides by $q \cdot q$.

$$2 \cdot (q \cdot q) = p \cdot p \tag{1}$$

Then $p \cdot p \in 2\mathbf{Z}$. Suppose further that $p \notin 2\mathbf{Z}$. Since $p \notin 2\mathbf{Z}$ implies $p \cdot p \notin 2\mathbf{Z}$, this cannot work: so $p \in 2\mathbf{Z}$ as well. Thus we may write $p = 2 \cdot r$ for some integer r. Substitute this into (1) to obtain

$$2 \cdot (q \cdot q) = (2 \cdot r) \cdot (2 \cdot r) = 4 \cdot (r \cdot r).$$

Cancel out a 2 on both sides, and we find that $q \cdot q \in 2\mathbf{Z}$. This rhymes with what happened with p. Following the logic,

$$q = 2 \cdot s$$

for some integer s. Note, however, that we have just witnessed two dividing both p and q. We assumed that p and q had no common factor, so the claim we supposed after the existence of x, that $x \in \mathbf{Q}$, must not hold. Either that, or x doesn't exist at all. $\square$

So, if we want numbers like x to exist, then we need a more extensive number system than the rationals. Why should x exist? Arguably, because we may search for it.

How To Find Square Roots

Both of the methods we present use the following fact about squaring positive rational numbers.

THEOREM 0.2 Let $x, y \in \mathbf{Q}^+$. Then $x < y$ exactly when $x \cdot x < y \cdot y$.

PROOF. Suppose $x < y$. Then $c \cdot x < c \cdot y$ for every positive rational number c. Thus

$$x \cdot x < x \cdot y = y \cdot x < y \cdot y,$$

where $c = x$ in the first inequality and $c = y$ in the second inequality. This shows that $x < y$ implies $x \cdot x < y \cdot y$. Also $x = y$ implies $x \cdot x = y \cdot y$, which when combined with the previous result yields $x \leq y$ implies $x \cdot x \leq y \cdot y$. Now suppose $x \geq y$. Then $y \leq x$, and by the second result, $y \cdot y \leq x \cdot x$, hence $x \cdot x \geq y \cdot y$. However, the statement "$x \geq y$ implies $x \cdot x \geq y \cdot y$" implies the statement "$x \cdot x < y \cdot y$ implies $x < y$, which completes the proof. □

Thus, both squaring and taking square roots preserves the order of the positive rationals. In other words, if we have a lower bound $a \cdot a$ on $x \cdot x$ where $a \in \mathbf{Q}$, and an upper bound $b \cdot b$ on $x \cdot x$ where $b \in \mathbf{Q}$, then

$$a \cdot a < x \cdot x < b \cdot b$$

implies $a < x < b$. Stricter bounds on $x \cdot x$ yield stricter bounds on x.

Binary Search

The following pseudocode details a method, known as *binary search,* which finds the square root of 2.

```
binsearch(a0, b0, epsilon):
    set x0 = (a0 + b0)/2, x = x0, (a1, b1) = x0^2 > 2 ? (a0, x0) : (x0, b0);
    set x1 = (a1 + b1)/2, x = x1;
    while(|xi - x(i-1)| >= epsilon):
        set (a(i+1), b(i+1)) = xi^2 > 2 ? (ai, xi) : (xi, bi);
        set x(i+1) = (a(i+1) + b(i+1))/2, x = x(i+1);
    return x;
```

Read the symbol-string "`X ? Y : Z`" as "`Y if X and otherwise Z`."

To get an idea of the method's speed, here are the values of `ai`, `bi`, `xi` for `binsearch(1, 2, 0.01)`:

ai	bi	xi	|xi - x(i-1)|
1	2	1.5	NULL
1	1.5	1.25	0.25
1.25	1.5	1.375	0.125
1.375	1.5	1.4375	0.0625
1.375	1.4375	1.40625	0.03125
1.40625	1.4375	1.4140625	0.0078125

For reference, $\sqrt{2} \approx 1.41421356237$.

THEOREM 0.3 (TRIANGLE INEQUALITY) Given $x, y \in \mathbf{Q}$,

$$|x + y| \leq |x| + |y|.$$

PROOF. Observe that

$$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y| \geq x^2 + y^2 + 2xy = (x + y)^2.$$

Taking square roots finishes the proof. □

Note that the term $|x_i - x_{i-1}|$ gets twice as small with each step. Further, as the index i grows larger, if $i < j$ then

$$\begin{aligned}
|x_i - x_j| &= |x_i + (0 + \cdots + 0) - x_j| \\
&= |x_i + ((x_{i-1} - x_{i-1}) + \cdots + (x_{j+1} - x_j)| \\
&= |(x_i - x_{i-1} + \cdots + (x_{j+1} - x_j)| \\
&\overset{!}{\leq} |x_i - x_{i-1}| + \cdots + |x_{j+1} - x_j| \\
&< |x_i - x_{i-1}|\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) \\
&= 2|x_i - x_{i-1}| \longrightarrow 0
\end{aligned}$$

Note that we use the triangle inequality in the ! step above.

Thus, the terms

$$x_0, x_1, x_2, \ldots$$

in the progression $(x_i)_i$ get closer and closer to each other as i gets large. This means that for very large i, we not only cannot tell the difference between x_i and x_{i+1}, we also cannot tell the difference x_i and $x_{i+1000000}$. Even simpler: far enough out, $(x_i)_i$ basically stops moving, thereby presenting striking evidence that $(x_i)_i$ actually zeros in on a point on our line, even if the number we would call that point happens to leave $\mathbf{Q}$.

So at least we have a candidate for $\sqrt{2}$. Note, however, that we have not yet shown that potentially distinct methods produce the same result.

Now, the binary search method of finding square roots works, but has its flaws: it crawls instead of darting to the target, and the iterates potentially dip below the target. Our best method will demonstrate wicked velocity, with the property that the iterates stay strictly above the target. It even has simpler pseudocode. The Greeks knew it (some say the Babylonians came up with it).

Hero's Method

We begin with a helpful fact:

> Theorem 0.4 (AM-GM Inequality) Given nonnegative rational numbers a and b,
>
> $$ab \leq \frac{(a+b)^2}{4}.$$

Proof. Observe that for every $x \in \mathbf{Q}$, either $x = 0$ or $x > 0$.

In particular, since $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$ we get

$$4ab \leq a^2 + 2ab + b^2 = (a+b)^2$$

from which the claim follows. □

We can use x_i and $\frac{2}{x_i}$ for a and b above.

For example, if $x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right)$, then this yields

$$2 = x_i \cdot \frac{2}{x_i} \leq \frac{1}{4}\left(x_i + \frac{2}{x_i}\right)^2 = x_{i+1}^2$$

so that as long as we initially have $x_0 > 0$, the squares of all subsequent terms never exceed 2.

Here follows the pseudocode for Hero's method.

```
hero(x0, epsilon):
    set x1 = (x0 + 2/x0)/2, x = x1;
    while(|xi - x(i-1)| >= epsilon):
        set x(i+1) = (xi + 2/xi)/2, x = x(i+1);
    return x
```

As promised, wicked velocity (reminder: $\sqrt{2} \approx 1.41421356237$):

xi	|xi - x(i-1)|
1	NULL
1.5	0.5
1.4167	0.0833...
1.41421569...	0.00245...
1.414213562...	0.000002...

A Comment On Convergence

Even though Hero's method outpaces binary search, the latter still has the property that for i large, x_i and $x_{i+1000000}$ amount to about the same difference as x_i and x_{i+1}. So the fact that Hero's method outpaces binary search means that the rate at which binary search and Hero's method match up directly relates to the rate at which binary search reaches the square root of 2 at all.

We also note that faster methods than Hero exist (search: Householder).

How Many Line Numbers?

We have shown that there are more line numbers than fractions. How many more?

Consider for a moment the collection of all solutions to

$$P(\xi) = 0$$

where P denotes any polynomial in ξ, all of whose coefficients live in $\mathbf{Q}$. We call such solutions **algebraic numbers.** A surprising fact we shall visit soon states that the algebraic numbers and $\mathbf{Q}$ have the same number of elements.

In other words, there exists a clever rearranging of the fractions so that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, ... all pair with points originally assigned to fractions, though in turn we lose certain properties: for example, the distance between 1 and 2 on the line need not match the length of the unit interval. (If you have not thought in these terms before, convince yourself that the same sort of thing can happen with $\mathbf{Z}$ and $2\mathbf{Z}$.) Yet some numbers leave even this collection (we call them *transcendental numbers*), such as π.

It gets weirder.

The number π belongs to a class of numbers called *computable numbers,* and the computable numbers have the same number of elements as $\mathbf{Q}$ as well. Yet some points on the line (actually, almost all, in a precise mathematical sense) correspond to numbers that leave the collection of computable numbers.

The Continuum Hypothesis

Philosophically speaking, we do not even have a clear idea of how many line numbers actually exist. In fact, the claim that would clarify matters (known as the *continuum hypothesis*) does not depend on the axioms of upon which we boot standard mathematics. In particular, both $\mathsf{CH} + \mathsf{ZFC}$ and $\text{-}\mathsf{CH} + \mathsf{ZFC}$ have been shown to exhibit logical consistency (said another way, neither system leads its users to contradiction).

A survey done by PhilPapers in 2020 asked its participants whether they thought the continuum hypothesis had a determinate truth value (as in, whether ZFC simply lacks the capacity to capture a fundamental truth about reality, or if no fact of the matter exists at all). The results divided into 38% saying determinate, 29% saying indeterminate, and 27% undecided.

Here we stand at the foot of the walls of consensus in human comprehension, having arrived here via a few innocuous queries about points on a line.

The Silver Lining

It seems that every time we reach an apex of crystallized understanding of the number line from its points, some random bit of chaos swoops in and shatters our progress. So, why study this number line object at all?

Because we can take the sheer uncountability of the number line and simply incorporate it into our toolkit. We can weaponize the very pathologies that afflicted us in ways that fortify our understanding instead of crippling it. Because while the number line may prove *algebraically* untenable, it happens to make *topological* sense.

So, forging bravely ahead, let $\mathbf{R}$ denote the set of numbers on the number line. If $x \in \mathbf{R}$ then we shall call x a **real number.**