Contents

Chapter 2

The Topological Perspective

2.1 Nearness

This section's introduction was adapted from Vickers' book "Topology via Logic."

When you measure a physical property that takes values in $\mathbf{R}$, the resulting measurement quantity is always some $r \in \mathbf{Q}$ up to some positive rational error $\varepsilon \in \mathbf{Q}^+$.

What the mesurement approximates, that is, the actual value, may in fact be some real x:

$$x \in (r - \varepsilon, r + \varepsilon).$$

However, if the ideal range for x is some possibly irrational interval U (where "ideal range" could mean, say, the most precise range of physically meaningful measurements) then the best we could do is

$$U \subseteq (r - \varepsilon, r + \varepsilon)$$

for increasingly narrow intervals.

For example, if one measurement leads us to believe that $U \subseteq (1, 2)$ but another says $U \subseteq (2, 3)$, then we do NOT have $U = 2$, but rather a defective system of measurement $U = \varnothing$, from which we may deduce nothing at all.

Thus, finite intersections either refine U or invalidate the existence of U altogether.

Suppose we wanted a precise description of U in terms of measurements we are actually able to make, i.e. rational intervals. We will prove in this chapter that $\mathbf{Q}$ is dense in $\mathbf{R}$.

Theorem 2.1. *Between any two* $a, b \in \mathbf{R}$ *such that* $a < b$*, there is* $c \in \mathbf{Q} \cap (a, b)$.

The boundary of U is exactly some (possibly irrational) distance δ away from x:

$$(x - \delta, x + \delta) \subset U.$$

Think of this interval as zooming in on a subset of U with a magnifier of some sort. If we can make arbitrarily precise rational measurements, then we may pick $r, \varepsilon \in \mathbf{Q}$ such that $x - \delta/2 < r < x$ and $x - r < \varepsilon < r - (x - \delta)$ and then (this takes some effort) we get:

$$x \in (r - \varepsilon, r + \varepsilon) \subseteq (x - \delta, x + \delta) \subseteq U$$

and in this way we may recover U as an aribtrary union of rational intervals.

Note that since x is always properly contained within $(r - \varepsilon, r + \varepsilon)$ and hence U, the value x may *never* lie on the boundary of U! Thus, we have a fairly precise description of U.

The interval U abstracts to the idea of an *open set*.

Open vs Closed

Definition 2.2. Let X be a set. A **topology** on X consists of a set of **open sets** $\tau \subseteq 2^X$ satisfying:

- $\varnothing \in \tau$ and $X \in \tau$
- τ is stable under finite intersections

$$((X_i)_{i=1}^n \subseteq \tau) \leq \left(\bigcap_{i=1}^n X_i \in \tau \right).$$

- τ is stable under arbitrary unions

$$((X_\iota)_{\iota \in J} \subseteq \tau) \leq \left(\bigcup_{\iota \in J} X_\iota \in \tau \right).$$

We call (X, τ) a **topological space.** Here, "stable" refers to the property that operating on the input wihtin a given setting produces an output that stays within that setting.

Every topological space also has a set of **closed sets** τ^c satisfying:

- $\varnothing \in \tau^c$ and $X \in \tau^c$
- τ^c is stable under finite unions
- τ^c is stable under arbitrary intersections

As the notation suggests, $(U \in \tau) = (U^c \in \tau^c)$.

Continuous Functions

Definition 2.3. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is **continuous** if

$$\forall Z \in 2^Y \left((Z \in \tau_Y) \leq (f^*(Z) \in \tau_X)\right)$$

i.e. if open sets on Y "pull back" to open sets on X.

Theorem 2.4. *The identity map is continuous.*

Proof. Follows from the fact that the domain of the identity map matches its codomain. □

Theorem 2.5. *Continuity is stable under composition.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ with U open in Z. Then

$$(g \circ f)^*(U) = (f^* \circ g^*)(U) = f^*(g^*(U))$$

which is open in X. □

The Category Top

Topological spaces form the objects of a category called Top, a subcategory of Set. The arrows in Top are continuous functions. As we observed, continuous functions are stable under composition and the identity map is always continuous.

Further, continuity itself may be viewed as a *functor*

$$\tau : \mathsf{Top}^{\mathsf{op}} \to \mathsf{Poset}$$

via $\tau(X, \tau_X) = (\tau_X, \subseteq)$ and $\tau(f : (X, \tau_X) \to (Y, \tau_Y)) = f^* : (\tau_Y, \subseteq) \to (\tau_X, \subseteq)$, where a functor

$$\mathcal{F} : \mathcal{C} \to \mathcal{D}$$

is a map that respects the arrow structure of $\mathcal{C}$ and $\mathcal{D}$. In particular, $(\cdot)^{\mathsf{op}}$ is the functor $\mathsf{Cat} \to \mathsf{Cat}$ which takes categories in the 2-category of categories and reverses all of their arrows.

Examples

Here are some objects/arrows in Top.

These examples are mostly of an abstractly topological sort; more concrete examples will follow shortly.

0. The empty space $(\varnothing, \{\varnothing\})$. There is a unique continuous map f_{init} going from the empty space into any other topological space (X, τ), wherein every preimage is empty: $(\varnothing, \{\varnothing\})$ is *initial* in Top.

1. The point space $(\{*\}, \{\varnothing, \{*\}\})$. There is a unique continuous map f_{fin} going from any other topological space (X, τ) into the point space, wherein the preimage of the point is all of X and the preimage of the empty set is empty: the point space is *final* in Top.

2. Let X be a set.

   a) The coarsest (fewest open sets) example of a topology on X is

   $$\tau_{\text{trivial}} = \{\varnothing, X\}.$$

   b) The finest (most open sets) example of a topology on X is

   $$\tau_{\text{discrete}} = 2^X.$$

3. Let X be a topological space and $\{Y_\iota\}_{\iota \in J}$ an indexed family of topological spaces.

   a) Suppose we have a family of functions

   $$f_\iota : X \to Y_\iota.$$

   The **initial topology** is the coarsest topology on X that makes the f_ι continuous.

   b) Now suppose we have a family of functions

   $$f_\iota : Y_\iota \to X.$$

   The **final topology** is the finest topology on X that makes the f_ι continuous.

4. Recall that $2 = \{0, 1\}$. The Sierpinski space $(2, \{\varnothing, \{1\}, 2\})$ is an example of a topological space whose topology is neither trivial nor discrete. The space acts as a *subobject classifier* in Top: any characterstic function χ_U from X into the Sierpinski space is continuous exactly when $U = {\chi_U}^*(\{1\})$.

5. A *metric* example which we will explore later, and perhaps the most tangible, is to consider **R** and let the most basic open set be some interval (a, b) with $a < b$, so that arbitrary open sets are then unions of these intervals. The resulting topology is called the **Euclidean topology** on **R**.

Projectors and Coprojectors

Example No. 3 above is paramount: several important spaces are constructed by considering initial and final topologies with respect to a family of functions.

Let X be a topological space.

Definition 2.6. A subset $Z \subseteq X$ can be viewed as a subspace of X when equipped with the **subspace topology,** which is initial with respect to the coprojector map $\iota : Z \to X$. Similarly, we may form a quotient $X/\sim$ from an equivalence relation $\sim$ defined on X, and this becomes a topological space when equipped with the **quotient topology,** which is final with respect to the projector map $\pi : X \to X/\sim$.

Now, let $\{X_\iota\}_{\iota \in J}$ be a collection of spaces.

Definition 2.7. If we want to treat each space X_ι as a subspace of a parent space $\Sigma = \coprod_{\iota \in J} X_\iota$, then taking the topology final with respect to the coprojectors $i : X_i \to \Sigma$ yields the **coproduct topology.** Similarly, if we want to treat each space X_ι as a quotient of a parent space $\Pi = \prod_{\iota \in J} X_\iota$, then taking the topology initial with respect to the projectors $p : \Pi \to X_p$ yields the **product topology.**

Connectedness and Compactness

Definition 2.8. A topological space (X, τ) is **connected** if there are no $X_1, X_2 \in \tau$ such that X can be written as $X_1 \cup X_2$ with $X_1 \cap X_2 = \varnothing$.

The continuous image of a connected set is another connected set.

Theorem 2.9. *Let* $f : X \to f_*(X)$ *be continuous. If* X *is connected then so is* $f_*(X)$.

Proof. By contraposition.

If $f_*(X)$ is disconnected then there exist open sets Y_1 and Y_2 such that $f_*(X) = Y_1 \cup Y_2$ with $Y_1 \cap Y_2 = \varnothing$.

Then
$$X = f^*(f_*(X)) = f^*(Y_1 \cup Y_2) = f^*(Y_1) \cup f^*(Y_2)$$
where $f^*(Y_1) \cap f^*(Y_2) = f^*(Y_1 \cap Y_2) = f^*(\varnothing) = \varnothing$, showing that X is disconnected.

Hence, if X is connected, then $f_*(X)$ must be connected. □

Definition 2.10. A topological space (X, τ) is **compact** if for every open cover of X there exists a finite subcover of X.

The continuous image of a compact set is another compact set.

Theorem 2.11. *Let* $f : X \to f_*(X)$ *be continuous. If* X *is compact then so is* $f_*(X)$.

Proof. Suppose that f is continuous and X is compact, and let $U = \{U_\iota\}_{\iota \in J}$ be an open cover of $f_*(X)$.

Since f is continuous, each $f^*(U_\iota)$ is open, and
$$X = f^*(f_*(X)) = f^* \left(\bigcup_\iota U_\iota \right) = \bigcup_\iota f^* \left(U_\iota \right),$$
i.e. $V = \{f^*(U_\iota)\}_{\iota \in J}$ is an open cover for X.

Since X is compact, we can reduce this to a finite subcover $V' = \{f^*(U_i)\}_{i=1}^n$. But then
$$f_*(X) = f_* \left(\bigcup_i f^*(U_i) \right) = \bigcup_i f_*(f^*(U_i)) = \bigcup_i U_i,$$
where the last equality follows since f is surjective.

Since we have found a finite subcover for $f_*(X)$, it follows that $f_*(X)$ must also be compact. □

Cost of Operations

To check if some subset is open or closed is relatively simple: just check if it's in the topology. That is, determining openness or closedness is $\mathcal{O}(1)$. To check if something is compact, however, you need to test a finite cover; to check if something is connected, you need to check an entire intersection. So both compactness and connetedness are $\mathcal{O}(n)$ to verify.

Now observe that computing a direct image is $\mathcal{O}(1)$, but unless you have inside information about the function, computing a preimage is $\mathcal{O}(n)$.

That is, checking openness/closedness is an easy peek because we already did the global work when we obtained the preimage map, but since direct image only pokes pointwise into the domain, it takes strong conditions such as connectedness or compactness to even make sense of global data via pushforward.

Neighborhoods and Points

Definition 2.12. Let X be a topological space, $S \subseteq X$.

A **neighborhood** of S is a subset V of X containing an open set U containing S:

$$S \subseteq U \subseteq V \subseteq X.$$

In particular, we are usually interested in the case when S is just a single point.

Consider a subset S of a topological space X.

- A point x is an **interior point** of S if S is a neighborhood of x. The set of all interior points of S is called the **interior** of S and is denoted S°.
- A point x is a **boundary point** of S if all neighborhoods of x contain at least one point in S and one point not in S. The set of all boundary points of S is called the **boundary** of S and is denoted ∂S.
- A point x is a **limit point** of S if all neighborhoods of x contain at least one point of S different from x itself. Note that a limit point of S does not have to be an element of S. The union of S with the set of all limit points of S is called the (topological) **closure** of S and is denoted $\overline{S}$.

Theorem 2.13. *Let S be a subset of a topological space X.*

Then S is open if and only if $S = S^\circ$, and S is closed if and only if $S = \overline{S}$.

Proof. Suppose S is open. Clearly we always have $S^\circ \subseteq S$, so it remains to show $S \subseteq S^\circ$. Let $x \in S$. Since S is open, S is a neighborhood of x:

$$x \in S \subseteq S \subseteq X.$$

So $x \in S^\circ$. Conversely, suppose $S \subseteq S^\circ$. Then every point of S is an interior point, that is, for every point $x \in S$ there exists an open set U_x such that $x \in U_x \subseteq S$. Then $\bigcup_{x \in S} U_x = S$, and since S is a union of open sets, S must itself be open.

Now suppose S is closed. Clearly we always have $S \subseteq \overline{S}$, so it remains to show $\overline{S} \subseteq S$. Let $x \in \overline{S}$. Then either $x \in S$ or x is a limit point of S. Suppose $x \notin S$ and x is a limit point of S. If S is closed, then S^c is open. If $x \notin S$, then necessarily $x \in S^c$. Note that S^c is a neighborhood of x by the same trick as before. However, since x is a limit point of S, we see S^c must contain a point of S (distinct from x), contradiction. So we must have $x \in S$.

Conversely, suppose every limit point of S is itself in S. Let $x \in S^c$. Since x is not in S, x is not a limit point of S. Thus it is not the case that all neighborhoods of x contain at least one point of S different from x itself. Thus there exists a neighborhood of x that does not contain any points of S, which amounts to there being an open set U such that $x \in U$ and such that U is contained within S^c. Thus S^c is a neighborhood of x, which means every point in S^c is an interior point, which means S^c is open, hence S is closed. □

Theorem 2.14. *Let X be a topological space, $S \subseteq X$.*

1. *Suppose $S' \subseteq S$. Then $(S')^\circ \subseteq S^\circ$.*
2. *Suppose $S \subseteq S''$. Then $\overline{S} \subseteq \overline{S''}$.*
3. *S° is the largest subset of S that is open in X.*
4. *$\overline{S}$ is the smallest set containing S that is closed in X.*

Proof.

1. Let $x \in (S')^\circ$. Then x is an interior point of S', i.e. there is some U_x open in X such that $x \in U_x \subseteq S'$. But $S' \subseteq S$, so we also have $x \in U_x \subseteq S$. Hence x is an interior point of S, i.e. $x \in S^\circ$.
2. Let $x \in \overline{S}$. Then x is a limit point of S. Suppose V is a neighborhood of x. Since x is a limit point, there is some $y \in V \cap S$ such that $y \neq x$. But $S \subseteq S''$, so $y \in V \cap S''$ and still $y \neq x$. Hence x is a limit point of S'', i.e. $x \in \overline{S''}$.
3. Let S' be open in X with $S' \subseteq S$. Then $S' = (S')^\circ \subseteq S^\circ$.
4. Let S'' be closed in X with $S \subseteq S''$. Then $\overline{S} \subseteq \overline{S''} = S''$.

□

Moore-Smith Convergence

Definition 2.15. Directed set.

Products of Compacts are Compact

One form of the Axiom of Choice is that a Cartesian product of nonempty sets is itself nonempty. Surprisingly, this fact is equivalent to something much stronger: a Cartesian product of compact topological spaces given the product topology is itself compact. This is due to how coarse the product topology is.

Three Separation Axioms

Definition 2.16. Let (X, τ) be a topological space. We say (X, τ) is...

- T_0 if for any $a, b \in X$ with $a \neq b$ there exists $U \in \tau$ such that
$$(a \in U) \quad \neq \quad (b \in U).$$
- T_1 if
$$p \in X \quad \leq \quad p^c \in \tau.$$
- T_2 or **Hausdorff** if for any $a, b \in X$ with $a \neq b$ there are $A, B \in 2^X$ such that
$$a \in A^\circ \ \wedge \ b \in B^\circ \quad \leq \quad A \cap B = \varnothing.$$

Summarizing:

points in T_0 spaces can be distinguised via open sets;
points in T_1 spaces are closed;
points in T_2 spaces may be separated by neighborhoods.

Theorem 2.17. *Every Hausdorff space is* T_1. *Every* T_1 *space is* T_0.

Proof. Let X be Hausdorff with $x \in X$. Suppose $y \in X \setminus x$. Then $y \neq x$, so by Hausdorff there exist $A, B \in 2^X$ such that $x \in A^\circ, y \in B^\circ, A \cap B = \varnothing$. Now observe that

$$y \in B^\circ \subseteq B \subseteq X \setminus A \subseteq X \setminus A^\circ \subset X \setminus x,$$

so every point in $X \setminus x$ is an interior point, i.e. $X \setminus x \in \tau$, so $x \in \tau^c$. Hence, all points of X are closed, i.e. X is T_1.

Now let X be T_1 with $x, y \in X$ such that $x \neq y$. Let $U = X \setminus x$. Then $U \in \tau$ via T_1. At the same time, $x \notin U$ and $y \in U$, which implies

$$(x \in U \vee y \in U) \wedge (x \notin U \vee y \notin U)$$

i.e. $(x \in U) \neq (y \in U)$, hence X is T_0. □

Now for some examples to complete the picture:

- Let (X, τ) be $\mathbf{R}$ with the cofinite topology, wherein $U \in \tau$ exactly when $X \setminus U$ is finite.

  Since $X \setminus x \in \tau$ for any $x \in X$ by definition, we have $x \in \tau^c$ for any $x \in X$, i.e. X is T_1. Pick $x, y \in X$ with $x \neq y$, and suppose there were $A, B \in 2^X$ such that $x \in A^\circ, y \in B^\circ$. Note that $A^\circ \in \tau$ and $B^\circ \in \tau$, so $X \setminus A^\circ$ and $X \setminus B^\circ$ are both finite, i.e. $(X \setminus A^\circ) \cup (X \setminus B^\circ)$ is also finite (finite unions of finite sets are finite), hence $X \setminus (A^\circ \cap B^\circ)$ is also finite by de Morgan's laws. But $\mathbf{R}$ is uncountable, which means $A^\circ \cap B^\circ$ is also uncountable, in particular $A^\circ \cap B^\circ$ is nonempty. And since $A^\circ \cap B^\circ \subset A \cap B$, it follows that $A \cap B$ is also nonempty. This implies that given any open neighborhood A of x and any open neighborhood B of y, we always have $A \cap B \neq \varnothing$. Hence, X is not Hausdorff.

- Let (X, τ) be the Sierpinski space, i.e. $(X, \tau) = (\{0, 1\}, \{\emptyset, \{1\}, \{0, 1\}\})$.

  Note that $\{1\}$ is an open set such that $0 \notin \{1\}$ but $1 \in \{1\}$, which implies $(0 \in \{1\} \wedge 1 \in \{1\}) \vee (0 \notin \{1\} \wedge 1 \notin \{1\})$, i.e. $(0 \in \{1\}) \neq (1 \in \{1\})$, so X is T_0. But by defininition, the closed sets of X are $\emptyset$, $\{0\}$, and X, i.e. the point 1 is not closed, so X is not T_1.

Theorem 2.18. *A compact subset of a Hausdorff space is closed.*

Proof. Let X be Hausdorff with $K \subseteq X$ compact. Take $x \in X \setminus K$. Then by Hausdorffness of X, for each y there exist disjoint open subsets U_y and V_y such that $x \in U_y$ and $y \in V_y$. The V_y form an open cover of K, so by compactness of K we may reduce this to a finite subcover $(V_i)_{i \in F}$ where $F \subseteq K$ finite. Let

$$U = \bigcup_{y \in F} U_y.$$

Then U is an open neighborhood of x disjoint from K. This implies that x is actually an interior point. Since an arbitrary point of $X \setminus K$ is interior, it follows that $X \setminus K$ is open. Hence, K is closed. □

Theorem 2.19. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof. Let X be compact, Y be Hausdorff, $f : X \to Y$ a continuous bijection with inverse $g : Y \to X$. Let $K \in X$ be closed. A closed subset of a compact space is itself compact. Thus K is compact, so $f_*(K) = g^*(K)$ is compact. A compact subset of a Hausdorff space is closed, so $g^*(K)$ is closed. This shows that the preimage of any arbitrary closed set K is also closed, so g is continuous. Hence, f has continuous inverse and is thus a homeomorphism. □

2.2 Distance

One thing you might pick up on from reading the last section is that, in principle, topology could have developed without the real numbers. However, that is not how the theory progressed. Rather, since mathematicians were aware of $\mathbf{R}$ while topologies were first specified, most of the theory is concentrated on how to leverage $\mathbf{R}$ to better understand topological spaces.

The category of metric spaces, Met, is the canonical example. Here, the lever is *distance*.

Definition 2.20. Let X be a set. A **distance** on X is a function

$$d : X \times X \to \mathbf{R}$$

satisfying the following properties:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

we call the ordered pair (X, d) a **metric space**.

One exception: when we do construct $\mathbf{R}$, our distance functions will be $\mathbf{Q}$–valued (to avoid circularity).

[add illustration: open ball, closed ball, sphere in the plane.]

Definition 2.21. Let (X, d) be a metric space. For $x \in X$ and $r \geq 0$, define:

- The **open ball** centered at x of radius r: $B(x, r) = \{p \in X : d(x, p) < r\}$
- The **closed ball** centered at x of radius r: $\overline{B}(x, r) = \{p \in X : d(x, p) \leq r\}$
- The **sphere** centered at x of radius r: $S(x, r) = \{p \in X : d(x, p) = r\}$

A mildly amusing phenomenon is that the closed ball isn't always properly contained within the closure of its corresponding open ball (though the opposite containment does always hold):

Theorem 2.22. *In general metric spaces we do not necessarily have* $\overline{B(x, r)} = \overline{B}(x, r)$.

Proof. Take $X = \{0\} \subseteq \mathbf{R}$ with $d(x, y) = |x - y|$. Then $\overline{B}(0, 0) = \{0\}$ but $\overline{B(0, 0)} = \overline{\varnothing} = \varnothing$. □

Theorem 2.23. *Metric spaces form topological spaces.*

Proof. Let (X, d) be a metric space. Now define an open set in X to be an arbitrary union of open balls:

$$U \in \tau \quad := \quad U = \bigcup_{\iota \in J} B(x_\iota, r_\iota).$$

We now check the axioms for a topological space.

- We have
$$\varnothing = \bigcup_{x \in X} B(x, 0) \qquad \text{and} \qquad X = \bigcup_{x \in X} B(x, 1).$$

- *Arbitrary unions of arbitrary unions of open balls form arbitrary unions of open balls:*
$$\bigcup_{\alpha \in A} \bigcup_{\iota \in J_\alpha} B(x_\iota, r_\iota) = \bigcup_{\iota \in \bigcup_{\alpha \in A} J_\alpha} B(x_\iota, r_\iota).$$

- *Finite intersections of arbitrary unions of open balls form arbitrary unions of open balls:*

  This is slightly trickier, but still doable. For shorthand, write $B_\iota = B(x_\iota, r_\iota)$. We note that the intersection of two open balls B_1 and B_2 can be written as an arbitrary union of open balls:
$$B_1 \cap B_2 = \bigcup_{p \in B_1 \cap B_2} B\left((p, \min(\{r_1 - d(x_1, p), r_2 - d(x_2, p)\})\right)$$

  Thus, the intersection of $U_A = \bigcup_{\alpha \in A} B_\alpha$ and $U_J = \bigcup_{\iota \in J} B_\iota$ can be written as follows:
$$\begin{aligned} U_A \cap U_J &= \left(\bigcup_{\alpha \in A} B_\alpha\right) \bigcap \left(\bigcup_{\iota \in J} B_\iota\right) = \bigcup_{\alpha \in A} \left(B_\alpha \cap \left(\bigcup_{\iota \in J} B_\iota\right)\right) \\ &= \bigcup_{\alpha \in A} \bigcup_{\iota \in J} (B_\alpha \cap B_\iota) = \bigcup_{(\alpha, \iota) \in A \times J} (B_\alpha \cap B_\iota) \\ &= \bigcup_{(\alpha, \iota) \in A \times J} \bigcup_{p \in B_\alpha \cap B_\iota} B\left(p, \min(\{r_\alpha - d(x_\alpha, p), r_\iota - d(x_\iota, p)\})\right) \\ &= \bigcup_{p \in \bigcup_{(\alpha, \iota) \in A \times J} B_\alpha \cap B_\iota} B\left(p, \min(\{r_\alpha - d(x_\alpha, p), r_\iota - d(x_\iota, p)\})\right). \end{aligned}$$

This completes the verification of the topological space axioms. □

The topology τ generated by these arbitrary unions of open balls is called the **metric topology.**

Theorem 2.24. *Metric spaces are Hausdorff.*

Proof. Let (X, d) be a metric space, $x, y \in X$ such that $x \neq y$, and let $r = d(x, y)$. Then

$$B(x, r/3) \cap B(y, r/3) = \varnothing$$

which shows that any pair of distinct points can be separated by neighborhoods. □

Continuous Functions

Since we have a distance function while working in Met, there are actually two competing definitions for what it means to be a continuous function: there is the ε–δ characterization, and then there is also openness of open preimages. In Top generally, these definitions need not coincide.

We now show that these two things are exactly the same in Met.

Theorem 2.25. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces with topologies* τ_X *and* τ_Y, $f : X \to Y$. *Then*

$$\forall x \in X \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x' \in X, \qquad d(x, x') < \delta \quad \leq \quad d(f(x), f(x')) < \varepsilon \tag{ε–δ}$$

exactly when

$$\forall Z \in 2^Y \left((Z \in \tau_Y) \leq (f^*(Z) \in \tau_X) \right) \tag{PRE}$$

Proof. (PRE) $\leq$ (ε–δ): Suppose (PRE). Let $x \in X$, $\varepsilon > 0$, $W = B(f(x), \varepsilon)$. Since $W \in \tau_Y$, $f^*(W) \in \tau_X$ by (PRE). So $f^*(W) = f^*(W)^\circ$, i.e. $B(x, r) \subseteq f^*(B(f(x), \varepsilon))$. Pick $\delta = r$. Let $x' \in X$.

$$\begin{aligned} d(x, x') < \delta &\leq x' \in B(x, \delta) \\ &\leq x' \in f^*(B(f(x), \varepsilon)) \\ &\leq f(x') \in B(f(x), \varepsilon) \\ &\leq d(f(x), f(x')) < \varepsilon. \end{aligned}$$

Hence, (ε–δ).

(ε–δ) $\leq$ (PRE): Suppose (ε–δ), and let $W \in \tau_Y$. Then $W = \bigcup_{\iota \in J} B(y_\iota, r_\iota)$, so

$$f^*(W) = \bigcup_{\iota \in J} f^*(B(y_\iota, r_\iota)).$$

[add an illustration here eventually]

Let $a \in f^*(B(y_\iota, r_\iota))$, $\varepsilon = r_\iota - d(f(a), y_\iota)$. Then there exists δ such that

$$B(a, \delta) \subseteq f^*(B(f(a), \varepsilon)) \subseteq f^*(B(y_\iota, r_\iota)).$$

But a was arbitrary, so $f^*(B(y_\iota, r_\iota)) \in \tau_X$, and by closure of open sets with respect to arbitrary unions,

$$f^*(W) \in \tau_X.$$

Hence, (PRE). □

Uniform and Lipschitz Continuity

Definition 2.26 (Uniform Continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

We recall that f is *pointwise continuous* (or simply **continuous**) if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x' \in X \qquad d_X(x, x') < \delta \quad \leq \quad d_Y(f(x), f(x')) < \varepsilon.$$

We say that f is **uniformly continuous** if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \quad \forall x' \in X \qquad d_X(x, x') < \delta \quad \leq \quad d_Y(f(x), f(x')) < \varepsilon.$$

Definition 2.27 (Lipschitz Continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

If there is some $c \geq 0$ such that

$$\forall x, x' \in X \qquad d_Y(f(x), f(x')) \leq c\, d_X(x, x')$$

then we say that f is **c–Lipschitz.** The constant c is known as the *Lipschitz constant* of f.

Theorem 2.28. *Every Lipschitz function is uniformly continuous.*

Proof. Pick $\delta = \varepsilon / c$. □

Example: Distance from a Set

Definition 2.29. Let (X, d) be a metric space and let $Z \subseteq X$.

$$d(x, Z) = \inf_{z \in Z} d(x, z).$$

Theorem 2.30. *Distance from a set is 1-Lipschitz.*

Proof. □

The Lebesgue Covering Lemma

Definition 2.31. Let (X, d) be a metric space, and let $\mathcal{O} = \{U_\iota\}_{\iota \in J}$ be an open cover of X.

A **Lebesgue number** is a number $\delta > 0$ such that:

every subset of X having diameter less than δ is contained in some $U_\iota \in \mathcal{O}$.

Theorem 2.32 (Lebesgue Covering Lemma)**.**

Let (X, d) be a compact metric space and let $\mathcal{O} = \{U_\iota\}_{\iota \in J}$ be an open cover of X.

Then $\mathcal{O}$ admits a Lebesgue number.

Proof. Since X is compact, we may reduce to some finite subcover $\mathcal{O}_{\mathsf{fin}}$.

Let $x \in X$. Then $x \in U^\circ = U$ for some $U \in \mathcal{O}_{\mathsf{fin}}$. Since ∂U is compact (closed subset of $\overline{U}$ bounded by X),

$$\min_{p \in \partial U} d(x, p)$$

is well-defined (by compactness) and positive (since $x \in U^\circ$). Note, however, that there could be multiple $U_i \in \mathcal{O}_{\mathsf{fin}}$ containing x, so

$$\max_{U_i \ni x} \min_{p \in \partial U} d(x, p)$$

is also well-defined (since the max is over a finite set) and positive (max of positive set). But X is compact, so

$$m = \min_{x \in X} \max_{U_i \ni x} \min_{p \in \partial U} d(x, p)$$

is also well-defined (by compactness of X) and positive (min of positive set). Let $\delta = m/2$.

Now suppose $z \in Z \subseteq X$ where $\operatorname{diam} Z = \inf_{z,z' \in Z} d(z, z') < \delta$. Then for $z' \in Z$,

$$\begin{aligned} d(z, z') \leq \operatorname{diam} Z < \delta &= \frac{1}{2} \min_{x \in X} \max_{U_i \ni x} \min_{p \in \partial U} d(x, p) \\ &\leq \frac{1}{2} \max_{U_i \ni z} \min_{p \in \partial U} d(z, p) \\ &= \frac{1}{2} \min_{p \in \partial U_{\mathsf{max}}} d(z, p) \end{aligned}$$

for some $U_{\mathsf{max}} \in \mathcal{O}_{\mathsf{fin}}$. Since everything defined in sight has been continuous, U_{max} doesn't jump around (though it may be nonunique). The above inequality asserts that any point z' away from z doesn't venture outside the boundary of some U_{max}. Hence $Z \subseteq U_{\mathsf{max}}$, which completes the proof. □

Continuity on Compacts is Uniform

Pointwise continuity is a local property of f, whereas uniform continuity is a global property of f.

Theorem 2.33 (Heine-Cantor)**.** *Let X and Y be metric spaces with $f : X \to Y$ a continuous function.*

Suppose X is compact. Then f is uniformly continuous.

Proof. Let $\varepsilon > 0$. Then $\varepsilon/2 > 0$ as well, so for every $x \in X$ there exists some $\delta_{x,\varepsilon/2}$ such that

$$f_*(B(x, \delta_{x,\varepsilon/2})) \subseteq B(f(x), \varepsilon/2).$$

Observe that

$$\mathcal{O}_\varepsilon = \{B(x, \delta_{x,\varepsilon/2}) : x \in X\}$$

forms an open cover of X. Since X is compact, $\mathcal{O}_\varepsilon$ admits a Lebesgue number λ. Pick $\delta = \lambda/2$.

Suppose $x_0, x_1 \in X$ with $d(x_0, x_1) < \delta$. Then $B(x_0, \delta)$ has diameter λ, so

$$B(x_0, \delta) \subseteq B(x, \delta_{x,\varepsilon/2})$$

for some $x \in X$, i.e. $B(x_0, \delta)$ is contained within some element of $\mathcal{O}_\varepsilon$. But then

$$f_*(B(x_0, \delta)) \subseteq f_*(B(x, \delta_{x,\varepsilon/2}) \subseteq B(f(x), \varepsilon/2)$$

so $f(x_0)$ and $f(x_1)$ are both contained within the ball $B(f(x), \varepsilon/2)$. But this ball has diameter ε, which implies

$$d(f(x_0), f(x_1)) < \varepsilon.$$

Hence pointwise continuity of f on X compact implies uniform continuity of f. □

2.3 Completeness

Completions: Unique up to Isometry

Theorem 2.34. *If a metric space completion exists, it is unique up to isometry.*

Proof. Let X be a metric space and let X^*, X^{**} be completions of X.

Every $x^* \in X^*$ has a Cauchy sequence $(x_n)_n \subset X$ such that $x_n \to x^*$ by completeness of X^*. But by completeness of X^{**}, we have $x_n \to x^{**}$ for some $x^{**} \in X^{**}$. Note that if $y_n \to x^*$ then $y_n - x_n \to 0$, hence $y_n \to x^{**}$; that is, the map $f : X^* \to X^{**}$ given by $f(x^*) = x^{**}$ is well defined. Note that $f(x) = x$ for all $x \in X$.

Now suppose $x_n \to x^*, y_n \to y^*$ in X^* and also $x_n \to x^{**}, y_n \to y^{**}$ in X^{**}. Then

$$\begin{aligned} d_*(x^*, y^*) &= \lim_{n\to\infty} d_*(x_n, y_n) \\ &= \lim_{n\to\infty} d(x_n, y_n) \\ &= \lim_{n\to\infty} d_{**}(x_n, y_n) = d_{**}(f(x^*), f(y^*)), \end{aligned}$$

so f is an isometry. □

This result can actually be strengthened to "unique up to unique isometry," though we won't do this.

Completions: Existence

Note: this proof depends on the existence and completeness of the real numbers **R**, a fact that we prove in the next section. Since we give that proof in full detail, we move somewhat briskly through the general case.

The proof is adapted from an argument given in Kolmogorov and Fomin.

Theorem 2.35. *Let (X, d) be a metric space. Then a completion of X exists.*

Proof. Declare two Cauchy sequences $(x_n)_n$ and $(y_n)_n$ in X to be *equivalent* if

$$\lim_n d(x_n, y_n) = 0,$$

and denote the set of equivalence classes by X^*. Define the distance between two points $x^*, y^* \in X^*$ via

$$d_*(x^*, y^*) = \lim_n d(x_n, y_n)$$

where $(x_n)_n$ is any representative of x^* and $(y_n)_n$ is any representative of y^*. This map is well-defined (i.e. does not depend on the choice of equivalence class representative) and is in fact a distance.

For $x \in X$, associate in X^* the equivalance class x^* of Cauchy sequences converging to x (for example, an easy to write down representative of this class would be $(x, x, x, \dots)$), so that when $x \in X$, we also have $x^* \in X$ via some nice representative, i.e. so that it makes sense to speak of $d\left(\lim_n x_n, \lim_n y_n\right)$ in the special case where $(x_n)_n$ and $(y_n)_n$ are constant sequences in X.

Recall that metrics respect convergence of sequences, i.e.

$$d\left(\lim_n x_n,\ \lim_n y_n\right) = \lim_n d(x_n, y_n).$$

But by definition,

$$\lim_n d(x_n, y_n) = d_*(x^*, y^*)$$

i.e. the map $(\cdot)^* : (X, d) \to (X^*, d_*)$ given by $x \mapsto x^*$ is an isometry! We now proceed as if $X \subset X^*$.

Now let $x^* \in X^*$ and $\varepsilon > 0$. Let $N \in \mathbf{N}$ be such that for $N < n, m$ we have $d(x_n, x_m) < \varepsilon$. Then

$$d(x_n, x^*) = \lim_m d(x_n, x_m) < \varepsilon$$

so $\overline{X} = X^*$; that is, X is dense in X^*.

By the above density result, given a Cauchy sequence $(x_n^*)_n \subset X^*$ there always exists a sequence $(x_n)_n \subset X$ equivalent to $(x_n^*)_n$ (simply pick x_n to be within $1/n$ of x_n^*). Since $x_n \to x^* \in X^*$, we have $x_n^* \to x^* \in X^*$ as well via equivalence of the two sequences, i.e. X^* is complete. □

The Banach Fixed-Point Theorem

Definition 2.36. Let (X, d) be a metric space, $f : X \to X$.

We say f forms a **contraction mapping** if there is some $\alpha \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$.

Theorem 2.37 (Banach fixed-point theorem)**.** *Let* (X, d) *be a nonempty complete metric space with a contraction mapping* $f : X \to X$. *Then* f *admits a unique fixed point* $x^* \in X$ *so that* $f(x^*) = x^*$.

Proof. Let $x_0 \in X$ and define

$$(x_n)_n = (x_0, A(x_0), A^2(x_0), \dots)$$

where $A^n(x) = A^{n-1}(A(x))$. For $n \leq m$,

$$d(x_n, x_m) = \alpha^n d(x_0, x_{m-n}) \leq \alpha^n \left(d(x_0, x_1) + \cdots + d(x_{m-n-1}, x_{m-n})\right) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1),$$

i.e. $(x_n)_n$ is Cauchy. By completeness of (X, d) we may write

$$x_n \to x_\infty$$

and by Lipschitz continuity of A (see below), $x_\infty = A(x_\infty)$. This proves existence of a fixed point x_∞.

To show uniqueness, note that

$$\big(A(x_\infty) = A(y_\infty)\big) \quad \leq \quad \big(d(x_\infty, y_\infty) \leq \alpha\, d(x_\infty, y_\infty)\big)$$

and since $0 \leq \alpha < 1$, we must have $x_\infty = y_\infty$. □

2.4 The Continuum

Theorem 2.38 (Construction of $\mathbf{R}$). *A completion of* $\mathbf{Q}$ *exists.*

Proof. Let $\mathbf{R} = \kappa(\mathbf{Q})/\mathfrak{m}_0$.

From previous work we know that $\mathbf{R}$ is an archimedean ordered field, which means that $\mathbf{Q} \subseteq \mathbf{R}$. We need to show that the completion of $\mathbf{Q}$ is $\mathbf{R}$ and that $\mathbf{R}$ is a complete metric space.

For elements $\alpha = (a_n)_n$, $\beta = (b_n)_n$ of $\kappa(\mathbf{Q})$, define

$$d_{\mathbf{R}}([\alpha], [\beta]) = [(d_{\mathbf{Q}}(a_n, b_n))_n].$$

By a previous proposition,

$$|d_{\mathbf{Q}}(a_m, b_m) - d_{\mathbf{Q}}(a_n, b_n)| \leq d_{\mathbf{Q}}(a_m, a_n) + d_{\mathbf{Q}}(b_m, b_n),$$

i.e. the sequence of distances is Cauchy because α and β are.

We claim that

$$(\alpha \sim \beta) = (d_{\mathbf{R}}([\alpha], [\beta]) = 0_{\mathbf{R}}).$$

Indeed, $\alpha \sim \beta$ if and only if for every $\varepsilon \in \mathbf{Q}^+$ there is some $N_\varepsilon \in \mathbf{Z}^+$ such that for all $n \geq N_\varepsilon$, $d_{\mathbf{Q}}(a_n, b_n) < \varepsilon$. But

$$(d_{\mathbf{Q}}(a_n, b_n) < \varepsilon) = (d_{\mathbf{Q}}(d_{\mathbf{Q}}(a_n, b_n), 0) < \varepsilon)$$

so the condition is equivalent to $d_{\mathbf{R}}([\alpha], [\beta]) = 0_{\mathbf{R}}$.

Another claim: the distance doesn't depend on equivalence class representative. This is easily shown if we know that $d_{\mathbf{R}}$ satisfies the triangle inequality, because then, if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, we have

$$d_{\mathbf{R}}([\alpha], [\beta]) \leq d_{\mathbf{R}}([\alpha], [\alpha']) + d_{\mathbf{R}}([\alpha'], [\beta']) + d_{\mathbf{R}}([\beta'], [\beta]) = d_{\mathbf{R}}([\alpha'], [\beta'])$$

$$d_{\mathbf{R}}([\alpha'],[\beta']) \leq d_{\mathbf{R}}([\alpha],[\alpha']) + d_{\mathbf{R}}([\alpha],[\beta]) + d_{\mathbf{R}}([\beta'],[\beta]) = d_{\mathbf{R}}([\alpha],[\beta])$$

showing that $d_{\mathbf{R}}([\alpha],[\beta]) = d_{\mathbf{R}}([\alpha'],[\beta'])$.

Let $\alpha = (a_n)_n,\ \beta = (b_n)_n,\ \gamma = (c_n)_n$ be in $\kappa(\mathbf{Q})$. Verifying the triangle inequality comes down to showing that

$$[(d_{\mathbf{Q}}(a_n, b_n))_n] \leq [(d_{\mathbf{Q}}(a_n, c_n) + d_{\mathbf{Q}}(c_n, b_n))_n].$$

Suppose equality does not hold. Then there exists an $\varepsilon \in \mathbf{Q}^+$ such that for all $N \in \mathbf{N}^{++}$ there is some $n_0 \geq N$ such that

$$d_{\mathbf{Q}}(d_{\mathbf{Q}}(a_{n_0}, c_{n_0}) + d_{\mathbf{Q}}(c_{n_0}, b_{n_0}), d_{\mathbf{Q}}(a_{n_0}, b_{n_0})) \geq \varepsilon$$

We may use the n_0 indices to build subsequences such that there is some $\varepsilon \in \mathbf{Q}^+$, $M \in \mathbf{Z}^+$ such that for all $m \geq M$,

$$d_{\mathbf{Q}}(a_m, b_m) < d_{\mathbf{Q}}(a_m, c_m) + d_{\mathbf{Q}}(c_m, b_m) + \varepsilon.$$

So the triangle inequality holds, which implies that the distance is well-defined.

Since $d_{\mathbf{R}}$ is built from $d_{\mathbf{Q}}$, it is both symmetric and nonnegative. We just checked the triangle inequality, and the fact that distance zero implies points are equal is a consequence of how we defined the quotient space $\hat{\kappa}(\Gamma)$. So $d_{\mathbf{R}}$ is a metric, which makes $(\mathbf{R}, d_{\mathbf{R}})$ a metric space.

Since real numbers are equivalence classes of Cauchy sequences, any open neighborhood of $[\alpha] \in \mathbf{R}$ will contain the tail of every sequence $(a_n)_n \in [\alpha]$, i.e. every open neighorhood of every point in $\mathbf{R}$ contains at least one rational. So $\mathbf{Q}$ is dense in $\mathbf{R}$.

We now show that $(\mathbf{R}, d_{\mathbf{R}})$ is complete.

Let $\Xi = (\xi_n)_n$ where $\xi_n = [(x_{(m,n)})_m]$ be a Cauchy sequence of real numbers. Then for every $\varepsilon \in \mathbf{Q}^+$ there is some $N_\varepsilon \in \mathbf{N}^{++}$ such that for all $n, n' \geq N_\varepsilon$ there is some $M_{n,n'} \in \mathbf{N}^{++}$ such that for all $m \geq M_{n,n'}$,

$$d_{\mathbf{Q}}(x_{(m,n)}, x_{(m,n')}) < \varepsilon.$$

Further, each ξ_n is Cauchy, so for every $n \in \mathbf{N}^{++}$ and $\varepsilon \in \mathbf{Q}^+$ there is some $M_{n,\varepsilon} \in \mathbf{N}^{++}$ such that for $m, m' \geq M_{n,\varepsilon}$, we have

$$d_{\mathbf{Q}}(x_{(m,n)}, x_{(m',n)}) < \varepsilon.$$

Since we have equivalence classes, we may work with convenient (i.e. rapidly converging) representatives from each class. That is, we may assume that for all $m, m', n \in \mathbf{Z}^+$, we have

$$d_{\mathbf{Q}}(x_{(m,n)}, x_{(m',n)}) < 2^{-\min(\{m,m'\})}.$$

Let $\varepsilon \in \mathbf{Q}^+$, and pick $N = \max(\{N_{\varepsilon/2}, \lceil 1 - \log_2(\varepsilon) \rceil\})$. so that

$$2^{-N} < 2^{-(1-\log_2(\varepsilon))} = \varepsilon/2.$$

Then

$$d_{\mathbf{Q}}(x_{(n,n)}, x_{(n',n')}) \leq d_{\mathbf{Q}}(x_{(n,n)}, x_{(n,n')}) + d_{\mathbf{Q}}(x_{(n,n')}, x_{(n',n')}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

so $\xi_\infty \in \mathbf{R}$. Further,

$$d_{\mathbf{Q}}(x_{(m,n)}, x_{(n,n)}) \leq d_{\mathbf{Q}}(x_{(m,n)}, x_{(m,m)}) + d_{\mathbf{Q}}(x_{(m,m)}, x_{(n,n)}) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so $\xi_n \to \xi_\infty$. This shows that $\mathbf{R}$ is complete, which ends the proof. □

Forms of Completeness

0. **Archimedean Property:** Let $\alpha, \beta \in \mathbf{R}$. Then $\exists N \in \mathbf{N}$ such that $\alpha < N\beta$.

1. **Cauchy Completeness:** Let $(x_i)_i \subset \mathbf{R}$. If $(x_i)_i$ is Cauchy, then $x_i \to x_\infty$ for some $x_\infty \in \mathbf{R}$.

2. **inf/sup Completeness:** Let $S \subseteq \mathbf{R}$ be nonempty.
If S is bounded above, then $\sup S$ exists. If S is bounded below, then $\inf S$ exists.

3. **Interval Refinement Property:** Let $([x_i - \epsilon_i, x_i + \epsilon_i])_i$ be a sequence of closed intervals such that
$$[x_0 - \epsilon_0, x_0 + \epsilon_0] \supset [x_1 - \epsilon_1, x_1 + \epsilon_1] \supset [x_2 - \epsilon_2, x_2 + \epsilon_2] \supset \cdots$$
with $\epsilon_i \to 0$. Then the intersection $\bigcap_i [x_i - \epsilon_i, x_i + \epsilon_i]$ is nonempty.

4. **Monotone Convergence Theorem:** Let $(x_i)_i \subset \mathbf{R}$. If $(x_i)_i$ is either strictly increasing and bounded above or strictly decreasing and bounded below, then $x_i \to x_\infty$ for some $x_\infty \in \mathbf{R}$.

5. **Bolzano-Weierstrass Theorem:** Let $(x_i)_i \subset \mathbf{R}$. If $(x_i)_i$ is bounded, then $\exists (x_{\sigma(i)})_i \subseteq (x_i)_i$ such that $x_{\sigma(i)} \to x_\infty$ for some $x_\infty \in \mathbf{R}$.

6. **Intermediate Value Theorem:** Let $a, b \in \mathbf{R}$ with $a < b$ and let $f : [a, b] \to \mathbf{R}$ be continuous. If $f(a) < 0 < f(b)$, then $f(c) = 0$ for some $c \in (a, b)$.

Theorem 2.39. *Suppose 0 holds. Then 1 through 6 are logically equivalent.*

Proof. We'll do $1 \le 2 \le 3 \le 1$ and then $2 \le 4 \le 5 \le 6 \le 2$.

$1 \le 2$ Suppose $S \subset \mathbf{R}$ is nonempty and bounded above. Let $s_0 \in S$ and let M_0 be an upper bound of S. Construct $(s_i)_i$ and $(M_i)_i$ as follows: if $\frac{s_i + M_i}{2}$ is an upper bound of S, set $(s_{i+1}, M_{i+1}) = (s_i, \frac{s_i+M_i}{2})$. Otherwise there is some $x \in S$ such that $x \ge \frac{s_i+M_i}{2}$, and in this case we set $(s_{i+1}, M_{i+1}) = (x, M_i)$. The sequence $(s_0, M_0, s_1, M_1, s_2, M_2, \dots)$ is Cauchy, since $|s_{i+1} - M_{i+1}| \le \frac{1}{2}|s_i - M_i|$. Let α be the limit of this sequence. Then $M_i \to \alpha$ and $s_i \to \alpha$. So α is an upper bound of S, for if there were some α' such that $\alpha < \alpha' < M_i$ for all $i \in \mathbf{N}$, then $\alpha' - \alpha$ would be an ϵ such that $|\alpha - M_i| > \epsilon$ for all $i \in \mathbf{N}$, which cannot happen since $M_i \to \alpha$. Furthermore, since $s_i \to \alpha$, for every $\epsilon > 0$ there is some $i \in \mathbf{N}$ such that $\alpha - s_i < \epsilon$, which is precisely the supremum condition.

$2 \le 3$ Suppose we have a sequence of nested closed intervals $([x_i - \epsilon_i, x_i + \epsilon_i])_i$ with $\epsilon_i \to 0$. The set $S = \overline{(x_i - \epsilon_i)_i}$ is nonempty and bounded above by $x_i + \epsilon_i$ for all $i \in \mathbf{N}$, and thus has a supremum. Let $\alpha = \sup S$. Then since α is an upper bound of S, $x_i - \epsilon_i \le \alpha$ for all $i \in \mathbf{N}$. But since α is the least upper bound, any upper bound will be at least α, in particular $\alpha \le x_i + \epsilon_i$ for all $i \in \mathbf{N}$. Combining these two facts, we see that $\alpha \in [x_i - \epsilon_i, x_i + \epsilon_i]$ for all $i \in \mathbf{N}$, i.e. the intersection $\bigcap_i [x_i - \epsilon_i, x_i + \epsilon_i]$ contains α and is thus nonempty.

$3 \leq 1$ Suppose $(x_i)_i \subset \mathbf{R}$ is Cauchy. Let $a_i = \min \overline{\{x_j : i < j\}}$, $b_i = \max \overline{\{x_j : i < j\}}$. Then for every $\epsilon > 0$ there is some $N \in \mathbf{N}$ such that for all $i \geq N$ we have $b_i - a_i < \epsilon$. In particular for every $i \in \mathbf{N}$ there is some $\sigma(i) \in \mathbf{N}$ such that $b_{\sigma(i)} - a_{\sigma(i)} < 2^{-i}$. We then have

$$[a_{\sigma(0)}, b_{\sigma(0)}] \supset [a_{\sigma(1)}, b_{\sigma(1)}] \supset [a_{\sigma(2)}, b_{\sigma(2)}] \supset \cdots$$

with $b_{\sigma(i)} - a_{\sigma(i)} \to 0$, so the intersection $\bigcap_i [a_{\sigma(i)}, b_{\sigma(i)}]$ is nonempty. Pick $x_\infty \in \bigcap_i [a_{\sigma(i)}, b_{\sigma(i)}]$. Now, $a_{\sigma(i)} < x_\infty < b_{\sigma(i)}$ for all $i \in \mathbf{N}$, so

$$0 < x_\infty - a_{\sigma(i)} < b_{\sigma(i)} - a_{\sigma(i)} \qquad \text{and} \qquad a_{\sigma(i)} - b_{\sigma(i)} < x_\infty - b_{\sigma(i)} < 0$$

for all $i \in \mathbf{N}$. By the squeeze theorem, we have both $a_i \to x_\infty$ and $b_i \to x_\infty$. At least one of $\{(a_{\sigma(i)})_i, (b_{\sigma(i)})\}$ attains infinitely many distinct values, let's say $(a_{\sigma(i)})_i$ does this. Construct the sequence $(c_i)_i$ to be $(a_{\sigma(i)})_i$ without repeats. This is a subsequence of both $(a_{\sigma(i)})_i$ and $(x_i)_i$. But $a_{\sigma(i)} \to x_\infty$, so $c_i \to x_\infty$, hence $x_i \to x_\infty$.

$2 \leq 4$ Let $(x_i)_i \subset \mathbf{R}$ be strictly increasing and bounded above. Then $\alpha = \sup_i x_i$ exists. Let $\epsilon > 0$. Then there is some x_i such that $\alpha < x_j + \epsilon$ for all $j > i$, hence $|\alpha - x_j| < \epsilon$ for all $j > i$, showing $x_j \to \alpha$.

$4 \leq 5$ Suppose $(x_i)_i \subset \mathbf{R}$ is bounded. Take the subsequence $(x_{\sigma(i)})_i = (\min\{x_j : j < i\})_i$. This sequence is strictly decreasing and bounded below, so it converges to some x_∞.

$5 \leq 6$ Let $f : [a, b] \to \mathbf{R}$ be continuous. Take the sequence $(x_i)_i$ that goes

$$a, \quad b, \quad \frac{a+b}{2}, \quad \frac{3a+b}{4}, \quad \frac{a+3b}{4}, \quad \frac{7a+b}{8}, \quad \frac{5a+3b}{8}, \quad \cdots$$

this sequence is dense in $[a, b]$. Since continuous functions map dense subsets to dense subsets, $f_*([a, b]) \cap [f(a), f(b)]$ forms a dense subset of $[f(a), f(b)]$. Construct the infinite array of subsequences by stipulating that row n of the array consists of the subsequence of $(x_i)_i$ for which $f(x_i) \in [f(a)/n, f(b)/n]$ for all terms in the $n - 1$th row (take row 0 to be $(x_i)_i$). By density, the rows of this infinite array are nonempty, in fact each row has infinitely many entries. Since the terms of the infinite array are bounded, it follows that the leftmost column of the array is bounded. Thus the leftmost column has a convergent subsequence. By sequential continuity of f, the image of this convergent subsequence is itself a convergent sequence. Due to how the infinite array of subsequences was constructed, though, the image of the limit of the convergent subsequence has no choice but to be 0. So the limit of the leftmost column sequence is a c such that $f(c) = 0$.

$6 \leq 2$ Let $S \subset \mathbf{R}$ be nonempty and bounded above and suppose that $\sup S$ does not exist. Let T be the set of all upper bounds of S, and define $f : \mathbf{R} \to \mathbf{R}$ to take the values -1 on $\mathbf{R} \setminus T$ and 1 on T. Then f is continuous, but for all $x \in \mathbf{R}$, we have $f \neq 0$, thereby falsifying the intermediate value theorem. So by contraposition, $\sup S$ must exist.

Hence, 1 through 6 are logically equivalent given 0. □